Tutorial 6

In the following problems, V denotes a finite-dimensional inner product space.

1. Show that for all linear operators $T \in \mathcal{L}(V)$ there exist normal operators $N_1, N_2 \in \mathcal{L}(V)$ such that $T = N_1 N_2$.
2. Suppose the only singular value of $T \in \mathcal{L}(V)$ is 1. What can you say about T?
3. Let $T \in \mathcal{L}(V)$ be a linear operator. Show that 0 is a singular value of T if and only if 0 is an eigenvalue of T.
4. (7.D.8) Let $R, S \in \mathcal{L}(V)$ be such that R is positive and S is an isometry. Define $T = SR$. Show that $R = \sqrt{T^*T}$.
5. Suppose $T \in \mathcal{L}(V)$ is a linear operator and $v \in V$ is such that $\|v\| = 1$. What is the largest possible value for $\|T(v)\|$?
6. Suppose $T \in \mathcal{L}(V)$ is such that for all $u, v \in V$, $\langle T(u), T(v) \rangle = 0$ if and only if $\langle u, v \rangle = 0$. Show there exists $c \in \mathbb{F}$ such that cT is an isometry.
7. Suppose $T \in \mathcal{L}(V)$ has singular value decomposition

   $$T(v) = \sum_{k=1}^{n} s_k \langle v, e_k \rangle f_k$$

   where $s_1, \dots, s_n \in \mathbb{F}$ are the singular values of T and $(e_1, \dots, e_n), (f_1, \dots, f_n)$ are orthonormal bases for V.

   Let $r \in \{1, \dots, n\}$ be such that $s_k \neq 0$ for $k \leq r$ and $s_k = 0$ for $k > r$. Define $T^+ \in \mathcal{L}(V)$ as

   $$T^+(v) = \sum_{k=1}^{r} \frac{1}{s_k} \langle v, f_k \rangle e_k$$

   The operator T^+ is called the *pseudoinverse of* T. What can you deduce about T^+?
8. Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$ is a linear operator with eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and singular values $s_1, \dots, s_n \in \mathbb{R}$. Show that

   $$|\lambda_1 \lambda_2 \cdots \lambda_n| = s_1 s_2 \cdots s_n$$